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OPTIMAL CONTROL OF THE ROTATION OF A SOLID WITH A FLEXIBLE ROD†

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Two optimal control problems may arise when a solid with a rigidly attached rod is rotating in a plane: how to steer the system from an initial phase state to a terminal state so as to minimize a quadratic cost functional, and time-optimal control. A new method is proposed for constructing optimal controls, based on the results of [1, 2] and methods of functional analysis. The controls are constructed as series in terms of a certain system of functions. Using the Voigt model of matter, some consideration is also given to a system with a viscoelastic rod and analogous results are obtained. The method is applicable to the problem of steering the system from an initial to a terminal phase state so as to minimize any convex functional of the control.

1. STATEMENT OF THE PROBLEM

WE WILL study a mechanical system consisting of a solid with a rigidly attached elastic rod of constant cross-section and mass uniformly distributed along its length. At the centre of mass of the solid we place an inertial system of coordinates $OX_1Y_1Z_1$, oriented so that the central axis of the rod lies in the $O_1X_1Y_1$ plane. The system may rotate about the O_1Z_1 axis, about which the torque $M'(t')$ of the controlling forces is applied. Attached to the solid is a system of coordinates $O'X'Y'Z'$, with its origin at the point of insertion of the rod, with the $O'X'$ axis pointing along the tangent to the neutral axis of the rod at the point of insertion and the $O'Z'$ axis parallel to the O_1Z_1 axis. The position of the entire system is uniquely described by the angle of deflection $\theta(t')$ (between the $O'X'$ and O_1X_1 axes) and the amount $y'(x', t')$ of transverse deformation of the rod at a point x' and time t' (Fig. 1).

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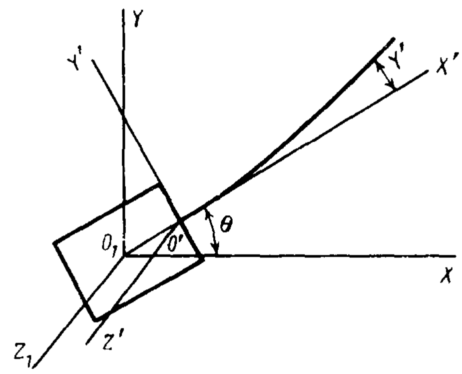

FIG. 1.

A mathematical model of our mechanical system is the following system of differential equations [3, 4]:

$$J\theta^{\cdot\cdot} + \int_0^1 (x+a)\, y_{tt}(x,t)\, dx = M(t) \tag{1.1}$$

$$y_{tt} + y_{xxxx} = -(x+a)\theta^{\cdot\cdot} \tag{1.2}$$

written in non-dimensional variables

$$x = x'/l, \quad y(x,t) = y'(x',t')/l, \quad t = bt', \quad b^2 = EI/(ml^4)$$
$$a = a'/l, \quad J = J'/(ml^3), \quad M(t) = M'(t')/(ml^3b^2)$$

with boundary conditions

$$y(0,t) = y_x(0,t) = 0, \quad y_{xx}(1,t) = y_{xxx}(1,t) = 0 \tag{1.3}$$

Here l, ET and m are the length, stiffness and mass per unit length of the rod, a' is the distance from the centre of mass of the solid to the point of insertion of the rod, and

$$J' = J_1' + m\int_0^l (x'+a')^2\, dx'$$

where J_1' is the moment of inertia of the body about the O_1Z_1 axis [below $J_1 = J_1'/(ml^3)$].

Henceforth $L_2(0, T)$ is the Hilbert space of square integrable functions $M(t)$ $(0 \leqslant t \leqslant T)$. The scalar product and norm in $L_2(0, t)$ are introduced as follows:

$$(M_1(t), M_2(t))_{L_2(0,T)} = \int_0^T M_1(t)\, M_2(t)\, dt, \; \|M(t)\|^2_{L_2(0,T)} = (M(t), M(t))_{L_2(0,T)} \tag{1.4}$$

We will consider the following optimal control problems.

Problem 1. Determine the control torque $M(t) \in L_2(0, T)$ that brings system (1.2), (1.2) from an initial state

$$\theta(0) = \theta_0, \quad \theta^{\cdot}(0) = \theta_0^{\cdot} \tag{1.5}$$

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_0^{\cdot}(x) \tag{1.6}$$

to a final state at a given time T

$$\theta(T) = \theta_T, \quad \theta^{\cdot}(T) = \theta_T^{\cdot}, \quad y(x,T) = y_T(x), \quad y_t(x,T) = y_T^{\cdot}(x) \tag{1.7}$$

and minimizes the functional

$$\Phi(M) = \|M(t)\|^2_{L_2(0,T)}/2 \tag{1.8}$$

No loss of generality is incurred by assuming that the starting time is zero. If $M(t)$ is discontinuous we will be interested in a generalized solution of problem (1.1)–(1.3).

Problem 2. Determine the control torque $M(t)\in L_2(0, T)$, $\Phi(M)\leqslant L<\infty$ that brings system (1.1)–(1.3) from state (1.5), (1.6) to state (1.7) in minimum time T.

2. A METHOD OF INTEGRATING THE BOUNDARY-VALUE PROBLEMS (1.1)–(1.3)

We will first subject the system of equations (1.1), (1.2) to a transformation based on ideas from [1, 2]. To that end we substitute $y_{tt}(x, t)$, as derived from (1.2) into (1.1), to get

$$\left[J-\int_0^1 (x+a)^2\,dx\right]\theta^{\cdot\cdot}-\int_0^1 (x+a)\,y_{xxxx}(x,t)\,dx=M(t) \tag{2.1}$$

Integrating the second integral in (2.1) by parts with due attention to the boundary conditions, we obtain an equation

$$J_1\theta^{\cdot\cdot}+ay_{xxx}(0,\ t)-y_{xx}(0,\ t)=M(t) \tag{2.2}$$

Now, isolating $\theta^{\cdot\cdot}$ from (2.2), we substitute it into the right-hand side of Eq (1.2). This gives the following equation for $y(x, t)$:

$$y_{tt}+y_{xxxx}-J_1^{-1}(x+a)(ay_{xxx}(0,\ t)-y_{xx}(0,\ t))=-J_1^{-1}(x+a)M(t) \tag{2.3}$$

with boundary conditions (1.3).

A solution of Eq. (2.3) satisfying the initial conditions (1.8) may be found by the Fourier method. We will first consider the homogeneous equation

$$y_{tt}+y_{xxxx}-J_1^{-1}(x+a)(ay_{xxx}(0,\ t)-y_{xx}(0,\ t))=0 \tag{2.4}$$

with boundary conditions (1.3). Writing the solution as $y(x, t) = v(x)\tau(t)$, we obtain a spectral boundary-value problem for $v(x)$:

$$v''''-J_1^{-1}(x+a)(av'''(0)-v''(0))=\lambda v \tag{2.5}$$

$$v(0)=v'(0)=0,\quad v''(1)=v'''(1)=0 \tag{2.6}$$

and an equation for $\tau(t)$:

$$\tau^{\cdot\cdot}+\lambda\tau=0 \tag{2.7}$$

The spectral problem (2.5)–(2.6) was studied in detail in [1], where a complete system of eigenvalues $0<\lambda_1<\lambda_2<\ldots<\lambda_n<\ldots$ and the corresponding eigenfunctions $v_n(x)$ ($n = 1, 2, \ldots$) were constructed. It was shown that $\lambda_n = \beta_n^4$, where β_n is the nth positive root of the characteristic equation

$$\begin{gathered}\operatorname{ch}\beta\cos\beta+1+J_1^{-1}\{2a\beta^{-2}\operatorname{sh}\beta\sin\beta+\\+\beta^{-3}[(a^2\beta^2+1)\operatorname{ch}\beta\sin\beta+(a^2\beta^2-1)\operatorname{sh}\beta\cos\beta]\}=0\end{gathered} \tag{2.8}$$

$$v_n(x)=v_n^*(x)/\langle v_n^*(x),\ v_n^*(x)\rangle$$

Here

$$\begin{gathered}v_n^*(x)=A_n\operatorname{ch}(\beta_n x)+B_n\operatorname{sh}(\beta_n x)+C_n\cos(\beta_n x)+D_n\sin(\beta_n x)+\\+(J_1\beta_n^2)^{-1}(x+a)(A_n-a\beta_nB_n+a\beta_nD_n-C_n)\end{gathered}$$

$$A_n=\operatorname{sh}\beta_n+[1-2a(J_1\beta_n^2)^{-1}]\sin\beta_n-2a^2(J_1\beta_n)^{-1}\cos\beta_n$$

$$B_n=-\operatorname{ch}\beta_n-[1+2a(J_1\beta_n^2)^{-1}]\cos\beta_n-2(J_1\beta_n^3)^{-1}\sin\beta_n$$

$$C_n=-[1+2a(J_1\beta_n^2)^{-1}]\operatorname{sh}\beta_n-\sin\beta_n-2a^2(J_1\beta_n)^{-1}\operatorname{ch}\beta_n$$

$$D_n=[1-2a(J_1\beta_n^2)^{-1}]\operatorname{ch}\beta_n+\cos\beta_n-2(J_1\beta_n^3)^{-1}\operatorname{sh}\beta_n$$

and the scalar product $\langle . , . \rangle$ is defined by

$$\langle v(x),\ w(x)\rangle=(v(x),\ w(x))_{L_2(0,1)}-J^{-1}(x+a,\ v(x))_{L_2(0,1)}(x+a,\ w(x))_{L_2(0,1)}$$

The functions $v_n(x)$ satisfy orthogoanlity conditions

$$\langle v_n(x),\ v_m(x)\rangle=\delta_{nm} \tag{2.9}$$

where δ_{nm} is the Kronecker delta.

Equation (2.7) with $\lambda = \lambda_n$ may be integrated in an obvious way, the result being

$$\tau_n(t)=a_n\cos(\omega_n t)+b_n\sin(\omega_n t)\quad(\omega_n=\beta_n^2)$$

Hence it follows that the solution of Eq. (2.4) satisfying the initial conditions (1.8) will be

$$y_0(x,t)=\sum_{n=1}^{\infty}v_n(x)(a_{0n}\cos(\omega_n t)+b_{0n}\omega_n^{-1}\sin(\omega_n t)) \tag{2.10}$$

$$a_{0n}=\langle y_0(x),\ v_n(x)\rangle,\quad b_{0n}=\langle y_0^{\cdot}(x),\ v_n(x)\rangle\quad(n=1,2,\ldots)$$

Let $Bv\equiv v''''$ denote the operator with domain

$$D(B)=\{v\,|\,v(x)\in W_2^4(0,1),\quad v(0)=v'(0)=v''(1)=v'''(1)=0\}$$

The operator B is selfadjoint and positive definite. We shall assume henceforth, that $y_0(x)\in D(B)$, $y_0^{\cdot}(x)\in D(B^{1/2})$, where $B^{1/2}$ is the positive root of B [5].

We will now construct a solution $y_*(x,t)$ of Eq. (2.3) that satisfies homogeneous initial conditions. To that end we express $x+a$ as a series in terms of the functions (2.8). Taking account of conditions (2.9), we have

$$x+a=\sum_{n=1}^{\infty}d_n v_n(x)$$

$$d_n=\langle x+a,\ v_n(x)\rangle=J_1J^{-1}(x+a,\ v_n(x))_{L_2(0,1)}=J_1J^{-1}c_n$$

Hence we clearly obtain

$$y_*(x,t)=-J^{-1}\sum_{n=1}^{\infty}v_n(x)c_n\int_0^t k_n(t-\tau)M(\tau)\,d\tau \tag{2.11}$$

where

$$k_n(t)=\omega_n^{-1}\sin(\omega_n t) \tag{2.12}$$

Now, substituting the solution $y(x,t)=y_0(x,t)+y_*(x,t)$ of Eq. (2.3) that satisfies conditions (1.6) into Eq. (1.1), we obtain a differential equation for $\theta(t)$:

$$\begin{gathered}\theta^{\cdot\cdot}(t)=qM(t)+\int_0^t G(t-\tau)M(\tau)\,d\tau+f(t)\\ q=1+J^{-1}\sum_{n=1}^{\infty}c_n^2,\quad G(t)=J^{-1}\sum_{n=1}^{\infty}c_n^2k_n^{\cdot\cdot}(t)\\ f(t)=\sum_{n=1}^{\infty}c_n\omega_n^2(a_{0n}\cos(\omega_n t)+b_{0n}\omega_n^{-1}\sin(\omega_n t)\end{gathered} \tag{2.13}$$

As a result, integrating (2.13), taking the initial conditions (1.5) into account, we obtain the desired solution:

$$\theta(t)=\theta_0+\theta_0^{\cdot}t+\int_0^t(t-t_1)\Big(qM(t_1)+\int_0^{t_1}G(t-\tau)M(\tau)\,d\tau+f(t_1)\Big)dt_1 \tag{2.14}$$

3. SOLUTION OF PROBLEM 1

In view of the relationships (1.7), (2.10) and (2.11), the results of Sec. 2 enable us to reformulate problem 1 as a smooth extremal problem with equality constraints: determine the minimum of the functional (1.8) subject to the constraints

$$\theta_T^{\cdot} = \theta_0^{\cdot} + \int_0^T \Big(qM(t_1) + \int_0^{t_1} G(t_1-\tau)M(\tau)\,d\tau + f(t_1)\Big)dt_1 \tag{3.1}$$

$$\theta_T = \theta_0 + \theta_0^{\cdot}T + \int_0^T (T-t_1)\Big(qM(t) + \int_0^{t_1} G(t_1-\tau)M(\tau)\,d\tau + f(t_1)\Big)dt_1 \tag{3.2}$$

$$a_{Tn} = a_{0n}\cos(\omega_n T) + b_{0n}\omega_n^{-1}\sin(\omega_n T) - J^{-1}c_n\int_0^T k_n(T-\tau)M(\tau)\,d\tau \tag{3.3}$$

$$b_{Tn} = -\omega_n a_{0n}\sin(\omega_n T) + b_{0n}\cos(\omega_n T) - J^{-1}c_n\int_0^T k_n^{\cdot}(T-\tau)M(\tau)\,d\tau \tag{3.4}$$

$$a_{Tn} = \langle y_T(x),\ v_n(x)\rangle, \quad b_{Tn} = \langle y_T^{\cdot}(x),\ v_n(x)\rangle \quad (n=1,\,2,\ldots)$$

Let $M_2(0,\ T)$ denote the set of functions $M(t)\in L_2(0,\ T)$ that satisfy conditions (3.1)–(3.4). A direct check shows that $M_2(0,\ T)$ is a closed convex set [6] and $\Phi(M)$ is a convex functional.

Hence, by the Kuhn–Tucker Theorem [6], a unique function $M^*(t)\in M_2(0,\ T)$ exists which minimizes the functional (1.8).

A few words about the practical calculation of $M^*(t)$. We first transform (3.1) and (3.2), taking (2.12), (2.13) and (3.3), (3.4) into consideration. It is at once clear that

$$\int_0^T \Big(qM(t_1) + \int_0^{t_1} G(t-\tau)M(\tau)\,d\tau\Big)dt_1 = \int_0^T \Big(qM(t_1) -$$
$$- J^{-1}\sum_{n=1}^{\infty} c_n^2\omega_n \int_0^{t_1} \sin(\omega_n(t_1-\tau))\,M(\tau)\,d\tau\Big)dt_1 =$$
$$= \int_0^T \Big[qM(t_1) - J^{-1}\sum_{n=1}^{\infty} c_n^2\omega_n\Big(\sin(\omega_n t_1)\int_0^{t_1}\cos(\omega_n\tau)M(\tau)\,d\tau -$$
$$- \cos(\omega_n t_1)\int_0^{t_1}\sin(\omega_n\tau)M(\tau)\,d\tau\Big)\Big]\,dt_1 = \int_0^T qM(t_1)\,dt_1 +$$
$$+ J^{-1}\sum_{n=1}^{\infty} c_n^2\Big(\cos(\omega_n T)\int_0^T \cos(\omega_n\tau)M(\tau)\,d\tau + \sin(\omega_n T)\int_0^T \sin(\omega_n\tau)M(\tau)\,d\tau -$$
$$- \int_0^T M(\tau)\,d\tau\Big) = \int_0^T M(t_1)\,dt_1 + J^{-1}\sum_{n=1}^{\infty} c_n^2\int_0^T k_n(T-\tau)M(\tau)\,d\tau$$

Then, in view of (3.3) and the equality

$$\int_0^T f(t_1)\,dt_1 = \sum_{n=1}^{\infty} c_n\,[\omega_n a_{0n}\sin(\omega_n T) + b_{0n}(1-\cos(\omega_n T))]$$

we can rewrite (3.1) in the form

$$l_{T0}(M) \equiv (1, M(t))_{L_2(0,T)} = A_0(T) \tag{3.5}$$

$$A_0(T) = \theta_T' - \theta_0' - \sum_{n=1}^{\infty} c_n (b_{Tn} - b_{0n})$$

A similar transformation is applied to (3.2).
As a result we obtain

$$l_{T1}(M) \equiv (T-t, M(t))_{L_2(0,T)} = A_1(T) \tag{3.6}$$

$$A_1(T) = \theta_T - \theta_0 - \theta' T - \sum_{n=1}^{\infty} c_n (a_{Tn} - a_{0n} - b_{0n} T)$$

It follows from (3.3) and (3.4), respectively, that

$$l_{T2n}(M) \equiv (k_n'(T-t), M(t))_{L_2(0,T)} = A_{2n}(T) \tag{3.7}$$

$$A_{2n}(T) = c_n^{-1} J(-b_{Tn} - \omega_n a_{0n} \sin(\omega_n T) + b_{0n} \cos(\omega_n T))$$

$$l_{T2n+1}(M) \equiv (k_n(T-t), M(t))_{L_2(0,T)} = A_{2n+1}(T) \tag{3.8}$$

$$A_{2n+1}(T) = c_n^{-1} J(-a_{Tn} + a_{0n} \cos(\omega_n T) + \omega_n^{-1} b_{0n} \sin(\omega_n T))$$

Remark 1. It follows from the properties of the functions $v_n(x)$ (as $n \to \infty$ they tend uniformly to the usual beam functions, which satisfy the boundary conditions (2.6) [1]) that $c_n \sim O(n^{-1})$, a_{0n}, $a_{Tn} \sim O(n^{-4})$, b_{0n}, $b_{Tn} \sim O(n^{-2})$ as $n \to \infty$.

Hence, also using the fact that $\omega_n \sim [\pi(2n+1)/2]^2$ as $n \to \infty$, we obtain $A_{2n}(T) \sim O(n^{-1})$, $A_{2n+1}(T) \sim O(n^{-3})$ as $n \to \infty$. Thus

$$\sum_{n=1}^{\infty} A_n^2(T) < \infty$$

Let $P_{2n+1}(0, T)$ denote the $(2N+4)$-dimensional subspace of $L_2(0, T)$ spanned by the following orthogonal system of functions in $L_2(0, T)$:

$$m_0(t) = T^{-1/2}, \quad m_{2j-1}(t) = T^{-1/2} \cdot 2 \cos(2\pi T^{-1} jt)$$

$$m_{2j}(t) = T^{-1/2} \cdot 2 \sin(2\pi T^{-1} jt) \quad (j = 1, \ldots, N)$$

Let us find the minimum of (1.8) over $P_{2n+1}(0, T)$ subject to conditions (3.5)–(3.8), where $n = 0, \ldots, N$, using Lagrange multipliers. The condition that the Lagrange function

$$L(M, \lambda_0, \ldots, \lambda_{2N+1}) = \Phi(M) + \sum_{j=0}^{2N+1} \lambda_j l_{Tj}(M)$$

be minimized under conditions (3.5)–(3.8) (in which $n = 1, \ldots, N$) yields a system of linear algebraic equations for the coefficients of the expansion of the function

$$M_N^*(t) = \sum_{j=0}^{2N} P_j m_j(t)$$

and the Lagrange multipliers $\lambda_0, \ldots, \lambda_{2N+1}$:

$$P_i + \sum_{j=0}^{2N+1} \lambda_j l_{Tj}(m_i) = 0 \quad (i = 0, \ldots, 2N) \tag{3.9}$$

$$\sum_{i=0}^{2N} P_i l_{Tj}(m_i) = 0 \quad (j = 0, \ldots, 2N+1) \tag{3.10}$$

which is uniquely solvable. Since the solution of the extremal problem (1.8), (3.5)–(3.8) is unique, $M_N^*(t)\to M^*(t)$ as $N\to\infty$.

This technique for constructing an optimal control $M^*(t)$ is valid for any convex functional $\Phi(M)$. Equations (3.9) may be non-linear in P_i, but system (3.9), (3.10) will still be uniquely solvable.

We will now consider another, more effective construction of an optimal control $M^*(t)$ for the functional (1.8). Define the system of functions

$$\varphi_0(t)=1,\quad \varphi_1(t)=T-t,\quad \varphi_{2j}(t)=k_j(T-t),\quad \varphi_{2j+1}(t)=k_j^{\cdot}(T-t) \qquad (j=1,2,\ldots) \tag{3.11}$$

Let $H_2(0, T)$ denote the subspace of $L_2(0, T)$ defined as the closed linear span of the functions (3.11).

The system (3.11) is not orthogonal in $L_2(0, T)$. However, we can apply the Schmidt orthogonalization procedure in $L_2(0, T)$ [5] to obtain a system of functions $\psi_n(t)$. We put

$$\begin{gathered}\psi_0\check{}(t)=\varphi_0(t),\quad \psi_0(t)=\psi_0\check{}(t)/\nu_0\\ \psi_1\check{}(t)=\varphi_1(t)-\alpha_{10}\psi_0(t),\quad \psi_1(t)=\psi_1\check{}(t)/\nu_1\\ \cdots\cdots\\ \psi_n\check{}(t)=\varphi_n(t)-\sum_{j=0}^{n-1}\alpha_{nj}\psi_j(t),\quad \psi_n(t)=\psi_n\check{}(t)/\nu_n\\ (\alpha_{nj}=(\varphi_n(t),\psi_j(t))_{L_2(0,T)},\quad \nu_n=\|\psi_n\check{}(t)\|_{L_2(0,T)})\end{gathered} \tag{3.12}$$

Together with (3.12) we introduce quantities $\beta_n(t)$ defined as follows:

$$\begin{gathered}\beta_0(T)=A_0(T)/\nu_0,\quad \beta_1(T)=(A_1(T)-\alpha_{10}\beta_0(T))/\nu_1,\ldots,\quad \beta_n(T)=\\ =\Big(A_n(T)-\sum_{j=0}^{n}\alpha_{nj}\beta_j(T)\Big)\Big/\nu_n\end{gathered} \tag{3.13}$$

A direct calculation shows that

$$\begin{gathered}\psi_0(t)=T^{-1/2},\quad \psi_1(t)=3^{1/2}T^{-1/2}-2\cdot 3^{1/2}T^{-3/2}t\\ \beta_0(T)=A_0(T)T^{-1/2},\quad \beta_1(T)=(2A_1(T)T^{-3/2}-A_0(T)T^{-1/2})3^{1/2}\end{gathered}$$

Remark 2. By Remark 2, the form of the functions (3.11) and the scheme used to construct (3.12) and (3.13),

$$\Theta(T)=\sum_{n=0}^{\infty}\beta_n^2(T)<\infty,\quad \lim_{T\to\infty}\Theta(T)=\infty,\quad \lim_{T\to\infty}\Theta(T)=0$$

The functions $\psi_n(t)$ form an orthonormal basis in $H_2(0, T)$. Hence, by (3.12) and (3.13), it follows that (3.5)–(3.8) are equivalent to the equalities

$$(M(t),\ \psi_n(t))_{L_2(0,T)}=\beta_n(T)\quad (n=0,1,\ldots) \tag{3.14}$$

Proposition 1. The solution of Problem 1 is given by the formula

$$M^*(t)=\sum_{n=0}^{\infty}\beta_n(T)\psi_n(t) \tag{3.15}$$

To prove this, we will represent $L_2(0, T)$ as a direct sum $L_2(0, T) = H_2(0, T)\oplus Q_2(0, T)$, where $Q_2(0, T)$ is the orthogonal complement of $H_2(0, T)$. It follows that any function $M(t)$ in the convex set $M_2(0, T)$ can be expressed as $M(t) = M^*(t)+Q(t)$, where $M^*(t)$ is given by (3.15) and $Q(t)$ is any function in $Q_2(0, T)$. Indeed, by (3.14) and (3.15),

$$(M(t),\ \psi_n(t))_{L_2(0,T)}=(M^*(t),\ \psi_n(t))_{L_2(0,T)}+(Q(t),\ \psi_n(t))_{L_2(0,T)}=\beta_n(T)$$

Since $\|M(t)\|^2_{L_2(0,T)}=\|M^*(t)\|^2_{L_2(0,T)}+\|Q(t)\|^2_{L_2(0,T)}$, it follows that

$$\inf_{M \in M_1(0,T)} \|M(t)\|^2_{L_2(0,T)} = \|M^*(\cdot)\|^2_{L_2(0,T)} + \inf_{Q \in Q_1(0,T)} \|Q(t)\|^2_{L_2(0,T)} = \\ = \|M^*(t)\|^2_{L_2(0,T)}$$

which proves the proposition.

4. THE SOLUTION OF PROBLEM 2

We introduce the function $\check{\Theta}(T) = \Theta(T) - L$. Let T^* denote the first positive root of the equation $\check{\Theta}(T) = 0$. Its existence follows from Remark 2.

Proposition 2. The pair $[T^*, M^*(t)]$, where $M^*(t)$ is given by formula (3.15) with $T = T^*$, is a solution of Problem 2.

The proof is an almost literal repetition of that of Proposition 1. One should only note that

$$\|M^*(t)\|^2_{L_2(0,T)} = \sum_{n=1}^{\infty} \beta_n^2(T)$$

5. THE CASE OF A VISCOELASTIC ROD

Consider the mechanical system described in Sec. 1, assuming now that the rod is viscoelastic. According to Voigt's rheological model of matter ($\sigma = E(\varepsilon + \nu\varepsilon^{\cdot})$) [7], the following system of differential equations is a model of the system:

$$J\theta^{\cdot\cdot} + \int_0^1 (x+a)\, y_{tt}(x,t)\, dx = M(t) \tag{5.1}$$

$$y_{tt} + h y_{txxxx} + y_{xxxx} = -(x+a)\theta^{\cdot\cdot} \tag{5.2}$$

with boundary conditions (1.3), where $h = \nu(EI/m)_{1/2} l^2$. The system is written in the same non-dimensional variables as in Sec. 1.

We will dwell on the construction of solutions of the boundary-value problem (1.3), (5.1) and (5.2) with initial conditions (1.5) and (1.6). By analogy with the discussion in Sec. 2, we transform the system of equations (5.1), (5.2), obtaining the following analogue of Eq. (2.3):

$$y_{tt} + h[y_{txxxx} - J_1^{-1}(x+a)(a y_{txxx}(0,t) - y_{txx}(0,t))] + \\ + y_{xxxx} - J_1^{-1}(x+a)(a y_{xxx}(0,t) - y_{xx}(0,t)) = -J_1^{-1}(x+a)M(t) \tag{5.3}$$

We will construct a solution of Eq. (5.3) with boundary and initial conditions (1.3) and (1.6), respectively. Using the Fourier method, we will first solve the equation

$$y_{tt} + h[y_{txxxx} - J_1^{-1}(x+a)(a y_{txxx}(0,t) - y_{txx}(0,t))] + \\ + y_{xxxx} - J_1^{-1}(x+a)(a y_{xxx}(0,t) - y_{xx}(0,t)) = 0 \tag{5.4}$$

Defining $y(x,t) = v(x)\tau(t)$, we see that $v(x)$ must satisfy the spectral boundary-value problem (2.5), (2.6), and $\tau(t)$ is a solution of the differential equation

$$\tau^{\cdot\cdot} + h\omega_n^2 \tau^{\cdot} + \omega_n^2 \tau = 0 \tag{5.5}$$

whose general solution is

$$\tau(t) = d_{n1}\exp(q_{n1}t) + d_{n2}\exp(q_{n2}t)$$

where $q_{n1,n2} = [-h\omega_n^2 \pm (h^2\omega_n^4 - 4\omega_n^2)^{1/2}]/2$. For those n for which q_{n1} and q_{n2} are complex, we take d_{n1} and d_{n2} to be complex conjugates.

Hence it follows that the solution of Eq. (5.4) satisfying the initial conditions (1.6) is

$$y_0(x, t) = \sum_{n=1}^{\infty} v_n(x)(d_{n1}\exp(q_{n1}t) + d_{n2}\exp(q_{n2}t))$$

$$d_{n1}=(a_{0n}q_{n2}-b_{0n})/(q_{n2}-q_{n1}), \quad d_{n2}=(a_0q_{n1}-b_{0n})/(q_{n1}-q_{n2})$$

and the solution of Eq. (5.3) is $y(x, t) = y_0(x, t) + y_*(x, t)$, where $y_*(x, t)$ is given by formula (2.11) with

$$k_n(t)=(\exp(q_{n1}t)-\exp(q_{n2}t))/(q_{n1}-q_{n2}) \tag{5.6}$$

The function $\theta(t)$ is determined by solving Eq. (2.13) with $k_n(t)$ as in (5.6) and

$$f(t) = -\sum_{n=1}^{\infty} c_n(d_{n1}q_{n1}^2\exp(q_{n1}t) + d_{n2}q_{n2}^2\exp(q_{n2}t))$$

All the results of Secs 3 and 4 hold for the boundary-value problem (1.3), (5.1) and (5.2). In this case

$$A_0(T) = \theta_T^{\cdot} - \theta_0^{\cdot} + \sum_{n=1}^{\infty} c_n(-b_{Tn} + d_{n1}q_{n1} + d_{n2}q_{n2})$$

$$A_1(T) = \theta_T - \theta_0 - \theta_0^{\cdot}T + \sum_{n=1}^{\infty} c_n[-a_{Tn} + (q_{n1}T + 1)d_{n1} + (q_{n2}T + 1)d_{n2}]$$

$$A_{2n}(T)=c_n^{-1}J(-b_{Tn}+d_{n1}q_{n1}\exp(q_{n1}T)+d_{n2}q_{n2}\exp(q_{n2}T))$$

$$A_{2n+1}(T)=c_n^{-1}J(-a_{Tn}+d_{n1}q_{n1}\exp(q_{n1}T)+d_{n2}q_{n2}\exp(q_{n2}T))$$

Example. Consider the following mechanical system: the solid is a cube of side $h = 1.5\times10^{-1}$ m; the rod is elastic, of length $l = 7.5\times10^{-1}$ m and square cross-section with side $h_1 = 10^{-2}$ m; the material is steel ($\rho = 7.8\times10^3$ kg/m^3 and $E = 2\times10^{11}$ N/m^2). As a result we obtain $J_1 = 3\times10^{-1}$; $J = 7.331\times10^{-1}$; $b = 2.55\times10^1$ sec^{-1}; $\beta_1 = 2.306$; $\beta_2 = 4.764$; $\beta_3 = 7.877$; $\beta_4 = 1.101\times10^1$; $\beta_5 = 1.414\times10^1$; $c_1 = 9.759\times10^{-1}$; $c_2 = 3.255\times10^{-1}$; $c_3 = 1.431\times10^{-1}$; $c_4 = 8.021\times10^{-2}$; $c_5 = 6.064\times10^{-2}$.

Figures 2, 3 and 4 are plots of the solutions $M(t)$ of Problem 1, and of the phase variable $\theta(t)$ (the dashed curve) for rotation of the system from the zero equilibrium position through an angle $\theta_0 = \pi/2$, with the vibrations of the rod completely damped out, for $T = 0.2$, 0.1 and 0.05 sec, respectively.

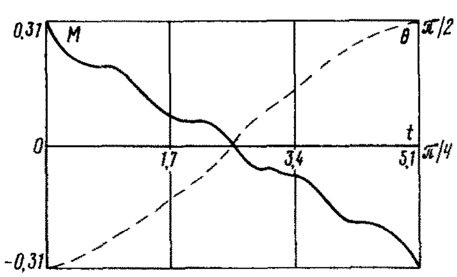

FIG. 2.

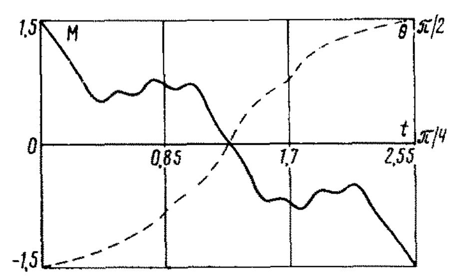

FIG. 3.

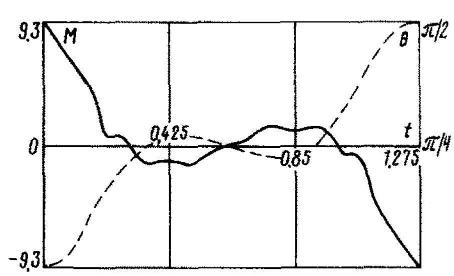

FIG. 4.

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SIGNAL PROPAGATION IN A RELATIVISTIC FLUID WITH VISCOSITY AND HEAT CONDUCTION†

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In order to eliminate the paradox due to the faster-than-light propagation of signals in standard relativistic models of fluids with dissipation, it is proposed to replace the dissipative coefficients in the constitutive equations by relaxation kernels, i.e. to use a theory with memory. It is shown that this yields signals with finite velocity, which, however, need not be less than that of light. The condition that the signal propagate at a velocity not exceeding that of light in a vacuum imposes certain *a priori* restrictions on the dissipative characteristics of a fluid.

INTRODUCTION

FLUIDS or gases with viscosity and heat conduction are described in relativity theory by two standard models, due respectively to Eckart [1] and Landau and Lifshits [2]; these models are physically equivalent [3], both preserving the characteristic feature of the non-relativistic Navier–Stokes–Fourier model, namely infinitely fast signal propagation in a locally attached inertial frame of reference (LAIFR).

In non-relativistic mechanics the unrealiability of theories with instantaneous signal propagation has long been recognized. A modified heat equation was proposed, with the result that the dynamics of the temperature field is governed by a telegraph-type equation. This idea was generalized by postulating a relaxation connection between the heat flux and the temperature gradient [5], i.e. basing the discussion on a theory of media with memory [6, 7]. A non-relativistic model of a viscous heat-conducting fluid with memory was constructed and it was proved that the signal velocity in such

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